April 15: Fibers of Ring Homomorphisms, continued

Fibers of Ring Homomorphisms

Last lecture we ended with the important corollary:

Corollary E4. Let $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}.l)$ be a flat local homomorphism of local rings. Suppose $\underline{x} = x_1, \ldots, x_r \in S$ have the property that their images in $S/\mathfrak{m}S$ form a regular sequence. Then $\underline{x}$ forms a regular sequence in S and $S/(\underline{x})S$ is flat over R.

Here is one of the main results from this section.

Theorem F4. Let $\phi : R \to S$ be a faithfully flat ring homomorphism.

(i) If S satisfies S_n, then R satisfies S_n.

(ii) If R satisfies S_n and the fibers $k(p) \otimes S$ satisfy S_n, for all $p \in \mathrm{Spec}(R)$, then S satisfies S_n.

(iii) Statements (i) and (ii) hold for Serre's condition R_n.

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Proof. For (i), take $P \in \mathrm{Spec}(R)$ and $Q \in \mathrm{Spec}(S)$ such that Q is minimal over PS. Then S_Q is faithfully flat over R_P and $\mathrm{height}(Q) = \mathrm{height}(P)$.

Now suppose $\mathrm{depth}(R_P) = r$. If $x_1, \ldots, x_r$ is a maximal regular sequence in R_P, by faithful flatness, these elements remain a regular sequence in S_Q. Moreover, there exists $c \in R$ with $Pc \in (x_1, \ldots, x_r)R$ and $c \notin (x_1, \ldots, x_r)R$.

Therefore, $PSc \in (x_1, \ldots, x_r)S$ and by flatness, $c \notin (x_1, \ldots, x_r)S$.

Since $Q^t \subseteq PS$ for some t, Q^t consists of zerodivisors modulo $(x_1, \ldots, x_r)S_Q$.

Thus, $\mathrm{depth}(S_Q) = \mathrm{depth}(R_P)$.

Since $\dim(R_P) = \dim(S_Q)$, it follows that if $\mathrm{depth}(S_Q) \geq \min\{n, \dim(S_Q)\}$,

then

$\mathrm{depth}(R_P) \geq \min\{n, \dim(R_P)\}$, and thus R satisfies S_n.

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For (ii) suppose R and the fibers $k(p) \otimes S$ satisfy S_n. Let $Q \subseteq S$ be a prime ideal, and set $P := Q \cap R$. We may localize at Q, so that R is a local ring with maximal ideal P and S is local at Q and flatness is preserved.

Note that S/PS is a fiber of our original ring homomorphism. Thus, R and S/P satisfy S_n.

Let $r := \mathrm{depth}(S/PS)$. Take $\underline{y} = y_1, \ldots, y_r \in S$ such that their images in S/PS form a regular sequence.

By Corollary E4, the sequence $\underline{y}$ is a regular sequence in S and $S/(\underline{y})S$ is flat over R.

Now take $x_1, \ldots, x_s \in R$, a maximal regular sequence so that $s := \mathrm{depth}(R)$.

Since $S/(\underline{y})S$ is flat over R, and $\underline{x} \cdot (S/(\underline{y})S) \neq S/(\underline{y})S$, the sequence $\underline{x}$ is a regular sequence on $S/(\underline{y})S$.

Thus, $\underline{y}, \underline{x}$ is a regular sequence in S.

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Therefore,

$$\mathrm{depth}(S) \geq \mathrm{depth}(R)+\mathrm{depth}(S/PS) \geq \min\{n, \dim(R)\}+\min\{n, \dim(S/PS)\}.$$

Consider the sum on the far right. If n is strictly less than one of $\dim(R)$ or $\dim(S/PS)$, then one of the terms in the sum equals n, so the sum, and hence $\mathrm{depth}(S)$ is greater than $\min\{n, \dim(S)\}$.

Suppose both $\dim(R)$ and $\dim(S/PS)$ are less than or equal to n.

The sum on the right above becomes $\dim(R) + \dim(S/PS) = \dim(S)$,

and thus $\mathrm{depth}(S) \geq \min\{n, \dim(S)\}$, in this case as well.

This shows S satisfies S_n.

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For part (iii), assume first that S satisfies R_n. Let $P \in \mathrm{Spec}(R)$ be a prime of height n or less and take $Q \in \mathrm{Spec}(S)$ minimal over PS. As before, $\mathrm{height}(Q) = \mathrm{height}(PS)$ and upon localizing at Q, we may assume that ϕ is a flat, local homomorphism between local rings of the same dimension, and S is a regular local ring.

Let k denote the residue field of R and take the start of a minimal free resolution

$$\cdots \to F_2 \to F_1 \to R \to k \to 0$$

of k as an R-module. It suffices to show $F_t = 0$, for some t, for then k will have finite projective dimension over R and thus, R will be a regular local ring.

Tensor this resolution with S. Since S is flat over R, the new sequence

$$\cdots \to F_2 \otimes S \to F_1 \otimes S \to S \to k \otimes S \to 0$$

is exact. Moreover, this is a minimal resolution over S.

Since S is regular, we must have $F_t \otimes S = 0$, for $t \geq \dim(S)$. Thus, some $F_t \otimes S = 0$, and by faithful flatness, $F_t = 0$, for some n. Thus, R is regular.

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Now suppose R and the fibers of ϕ satisfy R_n.

As before, we take $Q \in \text{Spec}(S)$ and localize S at Q, so that for $P = Q \cap R$, R is local at P, and hence regular, and the closed fiber S/PS is regular.

Now, P is generated by a regular sequence in R, which remains regular in S, by flatness. Moreover, Q/PS is generated by a regular sequence.

Putting these sequences together shows that Q is generated by a regular sequence which means S is a regular local ring, which is what we want. $\square$

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Here are some immediate corollaries.

Corollary G4. Let $\phi : R \to S$ be a faithfully flat ring homomorphism. Then:

(i) S is Cohen-Macaulay if and only if R is Cohen-Macaulay and the fibers $k(p) \otimes S$ are Cohen-Macaulay, for all $p \in \mathrm{Spec}(R)$.

(ii) S is regular if and only if R is regular and the fibers $k(p) \otimes S$ are regular, for all $p \in \mathrm{Spec}(R)$.

Proof. Immediate from Theorem F4. □

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Corollary H4. Let $(R, \mathfrak{m})$ be a local ring. Then:

(i) $\widehat{R}$ is reduced if and only R is reduced and its formal fibers are reduced. In particular, if R is a local domain, then R is analytically unramified if and only the formal fibers of R satisfy S_1 and R_0.

(ii) $\widehat{R}$ is integrally closed if and only if R is integrally closed and the formal fibers of R are integrally closed. In particular, if R is an integrally closed local domain, then $\widehat{R}$ is an integrally closed domain if and only if the formal fibers of R satisfy S_2 and R_1.

Proof. This follows from Theorem F4 and the characterizations of the reduced and integrally closed properties in terms of the Serre conditions S_n and R_n. □

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We now state some crucial elements in the definition of an excellent local ring.

Definitions. (i) Let k be a field and A an algebra over k (typically Noetherian). A is said to be geometrically regular if for every finite field extension $k \subseteq k'$, $k' \otimes_k A$ is regular.

(ii) The Noetherian ring R is said to be a G-ring if, for every $Q \in \mathrm{Spec}(R)$, the formal fibers of R_Q are geometrically regular.

As we are not going to prove anything of substance with these properties, a number of comments are in order.

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Comments. (i) If the k-algebra is geometrically regular, it is clearly regular. On the other hand, it turns out that if A is regular, and k' is a finite separable extension of k, then $k' \otimes_k A$ is automatically regular.

Thus if k is a perfect field, any regular k-algebra is geometrically regular. In particular, if k has characteristic zero, then any regular k-algebra is geometrically regular. Therefore, if R contains a field of characteristic zero, then R is a G-ring if and only if the formal fibers of R_Q are regular, for all $Q \in \mathrm{Spec}(R)$.

(ii) Suppose $(R, \mathfrak{m})$ is a local ring. To say that the formal fibers of R are geometrically regular, means that for every prime $p \in \mathrm{Spec}(R)$, the $k(p)$-algebra $k(p) \otimes_R \widehat{R}$ is geometrically regular. For non-local R, to be a G-ring means this property holds for R_Q, for all $Q \in \mathrm{Spec}(R)$.

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(iii) Some deep theorems concerning G-rings are:

(a) If $(R, \mathfrak{m})$ is a local ring and the formal fibers of R are geometrically regular, then R is a G-ring.In other words, in the local case, one does not have to check the formal fibers of R_Q, for $Q \in \mathrm{Spec}(R)$.

(b) A complete local ring is a G-ring.

(c) If R is a G-ring, then any finitely generated R-algebra is also a G-ring.

The proofs of these theorems involve a lot of machinery, including modules of differentials and the notions pertaining to *formal smoothness*.

Most of the details can be found in Matsumura's first book *Commutative Algebra*. As one might suspect, the difficulties mainly lie in the characteristic $p > 0$ case or the mixed characteristic case.

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(iv) A local ring is excellent (finally!) if it is a universally catenary G-ring.

When R is not local, an additional condition is required for a ring to be excellent, namely that, for any finitely generated R-algebra T, the set of primes $Q \in \text{Spec}(T)$ such that R_Q is regular form an open subset of $\text{Spect}(T)$.

It turns out that this condition holds automatically in a local G-ring, though this is also difficult to prove.

(v) An excellent local domain is a Nagata domain. This will follow from the theorem below.

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(vi) A finitely generated algebra over an excellent ring is excellent. The difficulty here lies in the transference of the G-ring property.

Also, homomorphic images and localizations of excellent rings are excellent.

(vii) It follows from what we have done this semester, and the statements above, that a regular local ring containing a field of characteristic zero is excellent.

We have already seen with Nagata's example that even a DVR in characteristic $p > 0$ need not be excellent.

There are examples of regular local rings of mixed characteristic that are not excellent.

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(viii) Standard examples of excellent rings include:

(a) Complete local rings (including fields).

(b) Characteristic zero Dedekind domains, including $\mathbb{Z}$.

(c) Finitely generated algebras over rings in (a), (b), and homomorphic images and localizations of these algebras.

(ix) A celebrated theorem of E. Kunz states that if R is a Noetherian ring containing a field of characteristic p, then R is excellent if R is a finite R^p-module.

R. Datta and K. Smith recently proved that if R is a domain containing a field of characteristic $p > 0$ and its quotient field K satisfies $[K : K^p] < \infty$, then if R is excellent, then R is finite over R^p.

We close this section with a theorem concerning Nagata rings and formal fibers. We need two preliminary results, the first of which we state as a Remark.

Remark. Let A be a ring , $S \subseteq A$ a multiplicatively closed ate. For A_S-modules M and N, we have $M \otimes_A N = M \otimes_{A_S} N$ and for A-modules M, N, we have $M_S \otimes_A N = (M \otimes_A N)_S$. We will use these properties below without comment.

The following lemma extends what we already know in the domain case to the case of reduced local rings.

Lemma I4. Let $(R, \mathfrak{m})$ be a reduced Nagata ring with total quotient ring K. Let T be a finite, integral extension of K such that T is also reduced. Let S denote the integral closure of R in T. Then S is a finite R-module.

Proof. Note that both K and T are direct sums of fields. Let $P \subseteq T$ be a minimal (and also, maximal) prime ideal. Then $K/(P \cap K) \subseteq T/P$ is a finite extension of fields.

Since $R/(P \cap R)$ is a Nagata ring, the integral closure of $R/(P \cap R)$ in T/P is a finite $R/(P \cap R)$-module. Since $S/(S \cap P)$ is contained in that integral closure, $S/(P \cap S)$ is a finite module over $R/(P \cap R)$, and hence, also a finite R-module.

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If we let $P_1, \ldots P_r$ denote the minimal primes of T, then $T = T/P_1 \oplus \cdots \oplus T/P_r$ (by the Chinese remainder theorem),

and we have

$$S \hookrightarrow S/(P_1 \cap S) \oplus \cdots \oplus S/(P_r \cap S) \subseteq T.$$

Since each $S/(P_i \cap S)$ is a finite R-module, S is a finite R-module. $\square$

Definition. Let k be a field. A k-algebra A is said to be geometrically reduced if $k' \otimes_k A$ is reduced for all finite extensions k' of k.

Here is the final theorem of this section.

Theorem J4. Let $(R, \mathfrak{m})$ be a local domain with quotient field K. Then R is a Nagata ring if and only if its formal fibers are geometrically reduced. In particular, if R is an excellent local domain, then R is a Nagata ring.

Proof. Assume that R is a Nagata ring and let $p \in \mathrm{Spec}(R)$. Set $U := R \backslash p$. We need to show that $k(p) \otimes)_R \widehat{R} = (\widehat{R}/p\widehat{R})_U$ is geometrically reduced.

Note that this is the generic formal fiber of R/p which is a Nagata ring. Thus, we may replace R/p by R and begin again assuming that R is a Nagata ring and show that its generic formal fiber $K \otimes \widehat{R} = \widehat{R}_U$ is geometrically reduced, where U is the set of non-zero elements of R.

Let K' be a finite field extension of K. We want $K' \otimes_K \widehat{R}_U$ to be reduced. Let $\tilde{K}$ denote the total quotient ring of $\widehat{R}$. We first note that

$$K' \otimes_K \widehat{R}_U \hookrightarrow K' \otimes_K \tilde{K},$$

since $K \subseteq K'$ is a faithfully flat extension of K-modules.

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Thus, it suffices to show $K' \otimes_K \tilde{K}$ is reduced. But

$$K' \otimes_K \tilde{K} = K' \otimes_R \tilde{K} \subseteq \mathrm{QR}(K' \otimes_R \widehat{R}),$$

where 'QR' denotes quotient ring.

Therefore, it suffices to show that $\mathrm{QR}(K' \otimes_R \widehat{R})$, and hence $K' \otimes_R \widehat{R}$ is reduced.

Now, let S denote the integral closure of R in K'. Then S is finite over R, so $\widehat{S} = S \otimes_R \widehat{R}$ and S is also a Nagata ring. Thus, $\widehat{S}$ is reduced, since a Nagata local domain is analytically unramified.

Therefore QR($\widehat{S}$) is reduced. On the other hand, we have

$$K' \otimes_R \widehat{R} = \mathrm{QR}(S) \otimes_R \widehat{R} \subseteq \mathrm{QR}(S \otimes_R \widehat{R}) = \mathrm{QR}(\widehat{S}),$$

which shows $K' \otimes_R \widehat{R}$ is reduced, which is what we want.

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Conversely, suppose the formal fibers of R are geometrically reduced. Let $p \in \mathrm{Spec}(R)$. We must show R/p satisfies N_2.

Since the generic formal fiber of R/p is geometrically reduced, as before, we may replace R/p by R, assume that R is a local domain with quotient field K and that the generic formal fiber $K \otimes_R \widehat{R} = \widehat{R}_U$ is geometrically reduced, where U is the set of non-zero elements in R.

Let K' be a finite field extension of K and S be the integral closure of R in K'.

We must show S is a finite R-module.

For this, it suffices to show that $S \otimes_R \widehat{R}$ is a finite $\widehat{R}$-module.

Note: we do not yet know $S \otimes_R \widehat{R} = \widehat{S}$.

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We have

$$S \otimes_R \widehat{R} \subseteq K' \otimes_R \widehat{R} \subseteq K' \otimes_R \widehat{R}_U = K' \otimes_K \widehat{R}_U,$$

with the last ring on the right being reduced by assumption.

Thus, $K' \otimes_R \widehat{R}$ is reduced and hence its quotient ring T is reduced.

On the other hand,

$$\widehat{R} \hookrightarrow \widehat{R}_U = K \otimes_R \widehat{R} \hookrightarrow K' \otimes_R \widehat{R}$$

which shows that $\widehat{R}$ and its quotient ring $\tilde{K}$ are reduced.

Since T is a finite extension of $\tilde{K}$, we are in the situation of Lemma I4 above.

$\widehat{R}$ is a reduced Nagata ring (since a complete local ring is a Nagata ring), so its integral closure in T is a finite $\widehat{R}$-module.

Since $S \otimes_R \widehat{R}$ is contained in this ring, $S \otimes_R \widehat{R}$ is a finite $\widehat{R}$-module, which completes the proof. □